

Analytic treatment of the excited instability spectra of the magnetically charged SU(2) Reissner-Nordström black holes

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The magnetically charged SU(2) Reissner-Nordström black-hole solutions of the coupled nonlinear Einstein-Yang-Mills field equations are known to be characterized by infinite spectra of unstable (imaginary) resonances $\{\omega_n(r_+, r_-)\}_{n=0}^{\infty}$ (here r_{\pm} are the black-hole horizon radii). Based on direct *numerical* computations of the black-hole instability spectra, it has recently been observed that the excited instability eigenvalues of the magnetically charged black holes exhibit a simple universal behavior. In particular, it was shown that the numerically computed instability eigenvalues of the magnetically charged black holes are characterized by the small frequency universal relation $\omega_n(r_+ - r_-) = \lambda_n$, where $\{\lambda_n\}$ are dimensionless constants which are independent of the black-hole parameters. In the present paper we study analytically the instability spectra of the magnetically charged SU(2) Reissner-Nordström black holes. In particular, we provide a rigorous *analytical* proof for the *numerically*-suggested universal behavior $\omega_n(r_+ - r_-) = \lambda_n$ in the small frequency $\omega_n r_+ \ll (r_+ - r_-)/r_+$ regime. Interestingly, it is shown that the excited black-hole resonances are characterized by the simple universal relation $\omega_{n+1}/\omega_n = e^{-2\pi/\sqrt{3}}$. Finally, we confirm our analytical results for the black-hole instability spectra with numerical computations.

I. INTRODUCTION

It is well known that the electrically charged U(1) Reissner-Nordström black-hole spacetime describes a *stable* solution of the coupled nonlinear Einstein-Maxwell field equations [1] (see also [2]). On the other hand, the magnetically charged SU(2) Reissner-Nordström black-hole spacetime [3] describes an *unstable* solution of the coupled nonlinear Einstein-Yang-Mills field equations [4–7]. In fact, the Reissner-Nordström black-hole solutions of the Einstein-Yang-Mills theory are characterized by an *infinite* spectrum of unstable perturbation modes. These unstable (exponentially growing in time) modes are described by an infinite set of imaginary black-hole resonances $\{\omega_n\}_{n=0}^{\infty}$ [4–7].

In a very interesting numerical investigation of the coupled Einstein-Yang-Mills equations, it was recently revealed by Rinne [8] that these unstable magnetically charged black-hole solutions play the role of approximate [9] codimension-two intermediate attractors (critical solutions [10]) in the nonlinear gravitational collapse of the Yang-Mills field [8, 11–14]. In particular, it has been shown explicitly [8] that the time spent in the vicinity of an unstable magnetically charged SU(2) Reissner-Nordström black-hole spacetime during a near-critical gravitational collapse of the nonlinear Yang-Mills field can be quantified by the characteristic scaling law [15]

$$\tau = \text{const} - \gamma \ln |p - p^*| . \quad (1)$$

It is interesting to note that the critical exponents in the scaling behavior (1), which characterizes the nonlinear near-critical gravitational collapse of the Yang-Mills field, are directly related to the imaginary eigenvalues which characterize the instability spectrum of the corresponding magnetically charged Reissner-Nordström black-hole spacetime [8]:

$$\gamma = 1/|\omega_{\text{instability}}| . \quad (2)$$

It is therefore physically interesting to investigate the characteristic instability (imaginary) resonance spectra $\{\omega_n(r_+, r_-)\}_{n=0}^{\infty}$ [16] of these magnetically charged black-hole solutions of the coupled nonlinear Einstein-Yang-Mills equations.

In his important numerical work, Rinne [8] has recently determined numerically the first three imaginary (unstable) resonant frequencies which characterize the magnetically charged SU(2) Reissner-Nordström black-hole spacetimes [17]. Subsequently, in [18] we have analyzed the detailed numerical data provided by Rinne [8] and revealed the intriguing fact that, to a good degree of accuracy, the numerically computed [8] excited instability eigenvalues of the magnetically charged SU(2) Reissner-Nordström black holes are characterized by the remarkably simple *universal* behavior

$$\omega_n(r_+ - r_-) = \lambda_n \quad \text{for} \quad \omega_n r_+ \ll (r_+ - r_-)/r_+ , \quad (3)$$

where $\{\lambda_n\}$ are dimensionless constants which seem to be *independent* of the black-hole parameters.

The main goal of the present paper is to determine *analytically* the characteristic instability spectra of the magnetically charged SU(2) Reissner-Nordström black-hole solutions of the coupled Einstein-Yang-Mills theory. In particular, in this paper we shall provide a rigorous analytical proof for the validity of the numerically suggested [8, 18] universal behavior (3) which characterizes the excited instability spectra of the SU(2) Reissner-Nordström black-hole spacetimes.

II. DESCRIPTION OF THE SYSTEM

The SU(2) Reissner-Nordström black-hole spacetime of mass M and unit magnetic charge is characterized by the spherically-symmetric line element [3]

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (4)$$

where the radially dependent mass function $m = m(r)$ is given by [19]

$$m(r) = M - \frac{1}{2r}. \quad (5)$$

The radii of the black-hole (outer and inner) horizons are given by

$$r_{\pm} = M \pm \sqrt{M^2 - 1}. \quad (6)$$

As shown in [20], linearized perturbation modes $\xi(r)e^{-i\omega t}$ [21] of the magnetically charged black-hole spacetime are governed by the wave equation

$$\left[\frac{d^2}{dx^2} + \omega^2 - U(x)\right]\xi = 0, \quad (7)$$

where the radial coordinate $x = x(r)$ is defined by the differential relation [22]

$$dx/dr = [1 - 2m(r)/r]^{-1}. \quad (8)$$

The effective radial potential which governs the Schrödinger-like wave equation (7) is given by [20]

$$U[x(r)] = -\frac{1}{r^2}\left[1 - \frac{2m(r)}{r}\right]. \quad (9)$$

It is worth emphasizing the fact that the radial function $U(x)$ in Eq. (7), which determines the spatial behavior of the black-hole perturbation modes, has the form of an effective binding potential. In particular, it is a negative definite function of the radial coordinate x and it vanishes asymptotically at the two boundaries $x \rightarrow \pm\infty$ of the magnetically charged black-hole spacetime. As shown in [20], well-behaved perturbation modes of the black-hole spacetime are characterized by spatially bounded (exponentially decaying) radial eigenfunctions at the two asymptotic boundaries:

$$\xi(x \rightarrow -\infty) \sim e^{|\omega|x} \rightarrow 0 \quad (10)$$

and

$$\xi(x \rightarrow \infty) \sim xe^{-|\omega|x} \rightarrow 0, \quad (11)$$

where $\omega = i|\omega|$.

The radial differential equation (7), supplemented by the physically motivated boundary conditions (10) and (11) [20], determine the discrete family $\{\omega_n(r_+, r_-)\}_{n=0}^{n=\infty}$ of unstable ($\Im\omega > 0$) resonances which characterize the SU(2) Reissner-Nordström black-hole spacetimes [5, 6]. Interestingly, below we shall show explicitly that the characteristic resonance spectrum of the magnetically charged black holes can be studied analytically in the regime $|\omega_n|r_+ \ll 1$ of small imaginary resonant frequencies. In particular, we shall derive a remarkably compact *analytical* formula [see Eq. (31) below] for the excited instability eigenvalues which characterize the SU(2) Reissner-Nordström black-hole solutions of the coupled Einstein-Yang-Mills theory.

III. THE CHARACTERISTIC RESONANCE CONDITION

The recent numerical results of Rinne [8] reveal that the excited instability resonances $\{\omega_n\}_{n=1}^{n=\infty}$ of the magnetically charged SU(2) Reissner-Nordström black-hole spacetimes are characterized by the property

$$|\omega_n|r_+ \ll 1 \quad ; \quad n = 1, 2, 3, \dots \quad (12)$$

As we shall now show, the Schrödinger-like differential equation (7), which governs the dynamics of the black-hole perturbation modes, is amenable to an *analytical* treatment in the regime (12) of *small* resonant frequencies.

It was pointed out in [7] that, in the $M \gg 1$ regime (the regime of weakly-magnetized SU(2) Reissner-Nordström black holes), the Schrödinger-like perturbation equation (7) can be transformed using the well-known Chandrasekhar transformations [23] to the physically equivalent Teukolsky-like radial equation [24]:

$$\Delta^2 \frac{d^2 \psi}{dr^2} + \left[\omega^2 r^4 + 2iM\omega r^2 - \Delta[2i\omega r + \ell(\ell+1)] \right] \psi = 0, \quad (13)$$

where the complex number

$$\ell \equiv \frac{-1 + i\sqrt{3}}{2} \quad (14)$$

plays the role of an effective spherical harmonic index (see [7] for details), and [24]

$$\Delta(r; M \gg 1) = r^2 - 2Mr. \quad (15)$$

The mathematical Chandrasekhar transformations [23] can also be used in the case of generic magnetically charged SU(2) Reissner-Nordström black holes [25], in which case the generalized expression for the radial function $\Delta(r)$ in the Teukolsky-like radial perturbation equation (13) is given by [24]

$$\Delta(r; M) = r^2 - 2Mr + 1. \quad (16)$$

It is convenient to use the dimensionless physical variables [26, 27]

$$z \equiv \frac{r - r_+}{r_+ - r_-} \quad ; \quad k \equiv -i\omega(r_+ - r_-) \quad ; \quad \varpi \equiv -i\omega \frac{2Mr_+}{r_+ - r_-}, \quad (17)$$

in terms of which the radial differential equation (13) reads

$$z^2(z+1)^2 \frac{d^2 \psi}{dz^2} + \left[-k^2 z^4 + 2kz^3 - \ell(\ell+1)z(z+1) - \varpi(2z+1) - \varpi^2 \right] \psi = 0. \quad (18)$$

The physically acceptable solution [28] of the radial perturbation equation (18) in the near-horizon $kz \ll 1$ region [29] can be expressed in terms of the familiar hypergeometric function [26, 27, 30]:

$$\psi(z) = z^{1+\varpi} (z+1)^{1-\varpi} {}_2F_1(-\ell+1, \ell+2; 2+2\varpi; -z). \quad (19)$$

The physically acceptable solution [31] of the radial perturbation equation (18) in the asymptotic region $z \gg \varpi + 1$ [32] can be expressed in terms of the familiar confluent hypergeometric function [26, 27, 30]:

$$\psi(z) = Ae^{kz} z^{\ell+1} {}_1F_1(\ell+2; 2\ell+2; -2kz) + Be^{kz} z^{-\ell} {}_1F_1(-\ell+1; -2\ell; -2kz), \quad (20)$$

For small resonant frequencies in the regime (12), the values of the dimensionless coefficients A and B in (20) can be determined by matching the two solutions [(19) and (20)] of the radial perturbation equation (18) in the overlap region [33]

$$\varpi + 1 \ll z \ll 1/k. \quad (21)$$

This matching procedure yields the expressions [26, 27]

$$A = \frac{\Gamma(2\ell+1)\Gamma(2+2\varpi)}{\Gamma(\ell+2)\Gamma(\ell+1+2\varpi)} \quad (22)$$

and

$$B = \frac{\Gamma(-2\ell-1)\Gamma(2+2\varpi)}{\Gamma(-\ell+1)\Gamma(-\ell+2\varpi)} \quad (23)$$

for the dimensionless coefficients $\{A, B\}$ in (20).

Substituting the expressions (22) and (23) into the far-region expression (20) of the radial eigenfunction and using the asymptotic ($z \gg 1$) properties of the confluent hypergeometric functions [30], one finds the large- z asymptotic behavior [26, 27]

$$\psi(z \rightarrow \infty) = \psi_1 r e^{-kz} + \psi_2 r^{-1} e^{kz} \quad (24)$$

of the radial eigenfunctions, where

$$(r_+ - r_-)\psi_1 = \frac{(2\ell+1)\Gamma^2(2\ell+1)\Gamma(2+2\varpi)}{\Gamma^2(\ell+2)\Gamma(\ell+1+2\varpi)}(-2k)^{-\ell} - \frac{(2\ell+1)\Gamma^2(-2\ell-1)\Gamma(2+2\varpi)}{\Gamma^2(-\ell+1)\Gamma(-\ell+2\varpi)}(-2k)^{\ell+1}, \quad (25)$$

and

$$(r_+ - r_-)^{-1}\psi_2 = \frac{(2\ell+1)\Gamma^2(2\ell+1)\Gamma(2+2\varpi)}{\ell(\ell+1)\Gamma^2(\ell)\Gamma(\ell+1+2\varpi)}(2k)^{-\ell-2} - \frac{(2\ell+1)(\ell+1)\Gamma^2(-2\ell-1)\Gamma(2+2\varpi)}{\ell\Gamma^2(-\ell)\Gamma(-\ell+2\varpi)}(2k)^{\ell-1}. \quad (26)$$

A spatially bounded (normalizable) radial eigenfunction which satisfies the physically motivated boundary condition (11) at spatial infinity [20] is characterized by the asymptotic relation $\psi(z \rightarrow \infty) \rightarrow 0$. Thus, the coefficient ψ_2 of the exploding exponent in the asymptotic expression (24) should vanish, yielding the characteristic resonance equation [see Eq. (26)]

$$(2k)^{2\ell+1} = \left[\frac{\Gamma(2\ell+1)\Gamma(-\ell)}{(\ell+1)\Gamma(-2\ell-1)\Gamma(\ell)} \right]^2 \frac{\Gamma(-\ell+2\varpi)}{\Gamma(\ell+1+2\varpi)} \quad (27)$$

for the instability eigenvalues of the SU(2) Reissner-Nordström black-hole spacetimes. Taking cognizance of Eq. (14), one can write the resonance equation (27) for the instability spectra of the magnetically charged black holes in the form [34]

$$k^{i\sqrt{3}} = 8^{i\sqrt{3}} e^{i2\pi/3} \frac{\Gamma^2(i\frac{\sqrt{3}}{2})}{\Gamma^2(-i\frac{\sqrt{3}}{2})} \frac{\Gamma(\frac{1-i\sqrt{3}}{2} + 2\varpi)}{\Gamma(\frac{1+i\sqrt{3}}{2} + 2\varpi)}. \quad (28)$$

IV. THE BLACK-HOLE EXCITED INSTABILITY SPECTRA

As we shall now show, the characteristic resonance equation (28) for the instability eigenvalues of the magnetically charged SU(2) Reissner-Nordström black holes can be solved analytically in the regime [35]

$$\varpi \ll 1. \quad (29)$$

In this small frequency regime the resonance equation (28) can be approximated by

$$k^{i\sqrt{3}} = 8^{i\sqrt{3}} e^{i2\pi/3} \frac{\Gamma^2(i\frac{\sqrt{3}}{2})}{\Gamma^2(-i\frac{\sqrt{3}}{2})} \frac{\Gamma(\frac{1-i\sqrt{3}}{2})}{\Gamma(\frac{1+i\sqrt{3}}{2})}, \quad (30)$$

which yields the characteristic *infinite* spectrum [36, 37]

$$\omega_n \times (r_+ - r_-) = i \times 8 e^{-\frac{2\pi}{\sqrt{3}}(n-\frac{1}{3}) + \frac{4\theta-2\phi}{\sqrt{3}}} ; \quad n = 1, 2, 3, \dots \quad (31)$$

of unstable ($\Im\omega > 0$) black-hole resonances, where

$$\theta \equiv \arg[\Gamma(i\sqrt{3}/2)] \quad ; \quad \phi \equiv \arg[\Gamma(\frac{1+i\sqrt{3}}{2})]. \quad (32)$$

It is worth emphasizing again that the analytically derived formula (31) for the characteristic instability spectra of the magnetically charged SU(2) Reissner-Nordström black holes is valid in the small frequency regime [see (17) and (29)]

$$\omega_n r_+ \ll \frac{r_+ - r_-}{r_+} . \quad (33)$$

This inequality implies that, for a given value of the black-hole dimensionless temperature $(r_+ - r_-)/r_+$, the analytical formula (31) describes an infinite family of unstable (imaginary) black-hole resonances in the regime

$$n \gtrsim \left| \ln\left(\frac{r_+ - r_-}{r_+}\right) \right| + 1 . \quad (34)$$

It is interesting to note that the instability spectra (31) of the magnetically charged SU(2) Reissner-Nordström black-hole spacetimes have the simple generic form $\omega_n \times (r_+ - r_-) = \text{constant}_n \equiv \lambda_n$ [see Eq. (3)]. We have therefore provided here an *analytical* proof for the *numerically*-observed [8, 18] universal behavior (3) of the black-hole excited instability eigenvalues.

V. NUMERICAL CONFIRMATION

It is of considerable physical interest to verify the validity of the analytically derived formula (31) for the excited instability eigenvalues of the SU(2) Reissner-Nordström black holes. The corresponding instability eigenvalues of the magnetically charged black holes were recently computed numerically in the very interesting work of Rinne [8]. In Table I we present the dimensionless ratio $\omega_2^{\text{ana}}/\omega_2^{\text{num}}$, where $\{\omega_2^{\text{ana}}(r_+)\}$ are the analytically calculated excited instability eigenvalues of the SU(2) Reissner-Nordström black holes as given by the analytical formula (31) and $\{\omega_2^{\text{num}}(r_+)\}$ are the numerically computed [8] instability eigenvalues of the magnetically charged black holes. From the data presented in Table I one finds a fairly good agreement between the analytically derived formula (31) for the excited instability spectra of the SU(2) Reissner-Nordström black holes and the corresponding numerically computed black-hole instability eigenvalues.

r_+	10.0	8.0	6.0	4.0	2.0	1.5
$\omega_2^{\text{ana}}/\omega_2^{\text{num}}$	1.046	1.025	1.011	1.005	1.007	1.012

TABLE I: The excited instability eigenvalues of the magnetically charged SU(2) Reissner-Nordström black holes. We display the dimensionless ratio $\omega_2^{\text{ana}}/\omega_2^{\text{num}}$, where $\{\omega_2^{\text{ana}}(r_+)\}$ are the analytically calculated instability eigenvalues of the SU(2) Reissner-Nordström black holes as given by the analytical formula (31) and $\{\omega_2^{\text{num}}(r_+)\}$ are the corresponding numerically computed [8] instability eigenvalues of the magnetically charged black holes. One finds a fairly good agreement between the numerical data of [8] and the analytically derived formula (31) for the excited instability spectra of the SU(2) Reissner-Nordström black holes.

VI. SUMMARY

The magnetically charged SU(2) Reissner-Nordström black-hole spacetimes describe a family of unstable solutions of the coupled nonlinear Einstein-Yang-Mills field equations [4, 5, 7, 8]. In particular, these magnetically charged black holes are known to be characterized by *infinite* spectra of imaginary (unstable) resonant frequencies $\{\omega_n(r_+, r_-)\}_{n=0}^{\infty}$. Based on direct *numerical* computations of the black-hole instability spectra [8], it has recently been pointed out [18] that the excited instability eigenvalues of the magnetically charged SU(2) Reissner-Nordström black holes are described, to a very good degree of accuracy, by the simple universal relation (3).

In the present paper we have studied analytically the characteristic instability spectra of the magnetically charged SU(2) Reissner-Nordström black-hole spacetimes in the small frequency regime. In particular, we have provided a simple *analytical* proof for the *numerically*-observed [8, 18] universal behavior [see Eq. (3)]

$$\omega_n(r_+ - r_-) = \lambda_n \quad \text{for} \quad \omega_n r_+ \ll (r_+ - r_-)/r_+ \quad (35)$$

which characterizes the black-hole excited instability resonances, where $\{\lambda_n\}$ are constants. Our analysis has revealed that these dimensionless constants (which are *independent* of the black-hole parameters) are given by the simple relation [see Eqs. (31) and (32)]

$$\lambda_n = i \times 8e^{-\frac{2\pi}{\sqrt{3}}(n-\frac{1}{3}) + \frac{4\theta-2\phi}{\sqrt{3}}} . \quad (36)$$

Finally, it is interesting to note that one finds from (31) the simple dimensionless ratio

$$\frac{\omega_{n+1}}{\omega_n} = e^{-2\pi/\sqrt{3}} \quad (37)$$

for the characteristic excited instability eigenvalues of the magnetically charged SU(2) Reissner-Nordström black holes. It is worth emphasizing the fact that the relation (37), which characterizes the instability resonance spectra of the magnetically charged black-hole spacetimes, is *universal* in the sense that it is *independent* of the physical parameters (masses and magnetic charges) of the SU(2) Reissner-Nordström black holes.

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 - [16] Here r_{\pm} are the black-hole horizon radii [see Eq. (6) below].
 - [17] As noted above, the SU(2) Reissner-Nordström black-hole solutions of the coupled nonlinear Einstein-Yang-Mills field equations are characterized by *infinite* spectra $\{\omega_n(r_+, r_-)\}_{n=0}^{\infty}$ of imaginary (unstable) resonant frequencies [6]. Reference [8] has provided, for the first time, detailed numerical results for the first three resonant frequencies which quantify the instability growth rates of these magnetically charged black-hole spacetimes.
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- [33] It is worth noting that the overlap radial region $\varpi \ll z \ll 1/k$ exists in the regime $|\omega|r_+ \ll 1$ of small resonant frequencies [see Eqs. (12) and (17)].
- [34] Here we have used Eq. 6.1.18 of [30].
- [35] Note that this regime corresponds to $\omega r_+ \ll (r_+ - r_-)/r_+$ [see Eq. (17)].
- [36] Here we have used the relation $1 = e^{-i2\pi n}$, where the integer $n = 1, 2, 3, \dots$ is the resonance parameter of the black-hole perturbation mode. In addition, we have used here Eq. 6.1.23 of [30].
- [37] As noted in [7], one finds the numerical ratio $\theta/2\phi = 1.0016$. Thus, one can replace, with an accuracy of 0.05%, the expression $(4\theta - 2\phi)/\sqrt{3}$ in (31) by the simpler term $\sqrt{3}\theta$.